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P-ADIC LINEAR GROUPS WITH ERGODIC AUTOMORPHISMS

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ABSTRACT

Let k be a locally compact, totally disconnected, nondiscrete field, and let G be a Lie group over k satisfying suitable conditions which depend on the characteristic of k. It is shown that G is compact if it admits a bicontinuous automorphism which is ergodic with respect to Haar measure.

Let G be a locally compact group and T a bicontinuous automorphism of G. The automorphism T is called ergodic if every Borel subset A of G such that $T(A) = A$ must either be null or have null complement with respect to Haar measure on G. It has been conjectured that only compact groups can possess ergodic automorphisms, and there is considerable supporting evidence for this conjecture. See, for example, [7] and the earlier references listed there. In particular, it was shown in [7] that if G is a Zariski-connected linear algebraic group defined over the locally compact, totally disconnected, nondiscrete field k such that the unipotent radical of G is defined over k and T is an automorphism of G also defined over k, then the group $G(k)$ of rational points of G over k is compact with respect to the usual locally compact topology if the restriction of T to $G(k)$ is ergodic. In this paper we shall extend this last result in several ways. Firstly, we shall show that $G(k)$ as above must be compact without the assumption that the ergodic automorphism T of $G(k)$ be the restriction of a k-automorphism of G. Moreover, $G(k)$ need not be connected and may be replaced by a sufficiently large subgroup of $G(k)$ (Theorem 2). In fact, when k is of characteristic zero, we may substitute for $G(k)$ any Lie group over k with a finite-dimensional, continuous representation over k whose kernel is, say, compact or solvable (Theorem 1).

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1. Preliminaries

Throughout this paper k is a nondiscrete, locally compact, totally disconnected field, $\bar{k}$ an algebraic closure of k, and $|\cdot|$ a multiplicative, discrete valuation on k. We also denote by $|\cdot|$ the canonical extension of this valuation to any finite extension of k.

Suppose G is a Zariski-connected linear algebraic group defined over k whose unipotent radical R_uG is also defined over k. We shall denote by G^* the subgroup of $G(k)$ generated by the subgroups $V(k)$, as V runs through the unipotent radicals of parabolic k-subgroups of G. The group G^* is normal in $G(k)$, and $G^* = \{e\}$ if G is reductive and anisotropic over k. When G is reductive, we shall also write G^+ for G^* to conform with the notation in [4].

For standard notation and results on Lie groups and linear algebraic groups see [6] and [2], [3], respectively.

We need the following two results, which appear as corollaries 1.3 and 1.7 in [7].

LEMMA 1. *Let G be a locally compact group which possesses an ergodic automorphism. If G is the union of an increasing sequence of compact open subgroups, then G is compact.*

LEMMA 2. *If the locally compact group G has a finite normal series of closed subgroups whose successive quotients are all unions of increasing sequences of compact open subgroups, then G is also the union of such an increasing sequence.*

LEMMA 3. *Let G be a σ-compact, locally compact, totally disconnected group with a closed, solvable, normal subgroup N. If the set of elements of G which lie in a compact subgroup is dense in G, then N is union of an increasing sequence of compact open subgroups.*

PROOF. Write

$$N = N^0 \supset N^1 \supset \cdots \supset N^n = \{e\},$$

where each N^j is a closed normal subgroup of G such that N^{j-1}/N^j is abelian, $1 \leqq j \leqq n$. Our lemma will follow from Lemma 2 if we show that each N^{j-1}/N^j is the union of an increasing sequence of compact open subgroups. Since N^{j-1}/N^j is abelian, totally disconnected and σ-compact, this is equivalent to showing that every element of N^{j-1}/N^j lies in a compact subgroup.

Fixing j and passing to the quotient group G/N^j we are reduced to considering the case where N is abelian, which we now assume. Let L be the set of elements of N which lie in a compact subgroup. Then L is an open subgroup of N which is normal in G. Thus $N' = N/L$ is a discrete, abelian, torsion-free, normal subgroup of $G' = G/L$. Let $x \in N'$. There exists a compact open subgroup H of G' such that $H \cap N' = \{e\}$ and every element of H centralizes x, and our hypothesis on G implies that for some $h \in H$, the product xh lies in a compact subgroup of G'. But then there exists a positive integer n such that $x^n h^n = (xh)^n \in H$, whence $x^n \in H \cap N' = \{e\}$. Thus $x = e$, and we have shown that $L = N$ as desired.

LEMMA 4. *Let $x \in GL_n(k)$. Then x is contained in a compact subgroup of $GL_n(k)$ if and only if all of its eigenvalues have valuation one.*

PROOF. If K is a finite extension of k, then $GL_n(k)$ is a closed subgroup of $GL_n(K)$. Hence we may replace k by a finite extension over which x may be put in upper triangular form, so let us assume x is upper triangular. Let B be the group of upper triangular matrices in $GL_n(k)$, let T and U be the subgroups of B consisting of all diagonal and of all unipotent upper triangular matrices, respectively, and let T_0 be the subgroup of T of all elements whose diagonal entries are units in the valuation ring of k. Then T_0 is compact, and U has a normal series of closed subgroups whose successive quotients are isomorphic to the additive group of k. Thus $T_0 \cdot U$ is the union of an increasing sequence of compact open subgroups by Lemma 2. If all of the eigenvalues of x have valuation one, then $x \in T_0 \cdot U$, so x lies in a compact subgroup of B. Conversely, T_0 is the largest compact subgroup of $T = B/U$. Thus if x is contained in a compact subgroup of B, then $x \in T_0 \cdot U$.

LEMMA 5. *The set of elements of $GL_n(k)$ which are contained in a compact subgroup is open and closed in $GL_n(k)$.*

PROOF. Let $x \in GL_n(k)$ and let λ be an eigenvalue of x. Then λ belongs to an extension K of degree n of k, so $|\lambda| = |N_{K/k}(\lambda)|^{1/n}$. Thus the set of valuations of eigenvalues of elements of $GL_n(k)$ is discrete. Furthermore, it is standard that if $P \in k[X]$ and Q is another element of $k[X]$ all of whose coefficients are sufficiently close in k to those of P, then the valuations of the roots of P and of Q are the same [1, ch. 6, §3]. It follows that the set of elements of $GL_n(k)$ whose eigenvalues have any prescribed valuations is open, and therefore also closed. Our lemma now follows from Lemma 4.

2. Ergodic automorphisms (k of characteristic zero)

Suppose that k is of characteristic zero, so that k is a finite separable extension of the field Q_p of p-adic numbers for some prime p. There is then a functor R_{k/Q_p} of "restriction of scalars" which maps the category of Lie groups over k to the category of p-adic Lie groups [5, 5.14], [6, p. 99]. If G is a Lie group over k, then $R_{k/Q_p}G$ is topologically isomorphic to G. Thus if G, G' are Lie groups over k, any continuous homomorphism $\rho: G \to G'$ may be viewed as a continuous homomorphism of $R_{k/Q_p}G$ into $R_{k/Q_p}G'$. If $G' = \boldsymbol{GL}_n(k)$, then $R_{k/Q_p}G'$ may be canonically identified to a closed subgroup of $\boldsymbol{GL}_{dn}(\boldsymbol{Q}_p)$, where $d = [k : \boldsymbol{Q}_p]$. Therefore if ρ is a continuous finite-dimensional representation of G over k, then it yields one of $R_{k/Q_p}G$ over $\boldsymbol{Q}_p$. We recall that if $k = \boldsymbol{Q}_p$, then every continuous homomorphism of Lie groups over k is analytic [6, III, §8, no. 1, th. 1] and every closed subgroup of a Lie group over k is a Lie group over k (loc. cit. no. 2, th. 2).

LEMMA 6. *Let k be of characteristic zero. Let G be a σ-compact Lie group over k and $\rho: G \to \boldsymbol{GL}_n(k)$ a faithful continuous representation of G over k. Suppose that the set of elements of G which are contained in a compact subgroup is dense in G. Then G is the union of an increasing sequence of compact open subgroups.*

PROOF. The above remarks show that the restriction of scalars reduces us to the case where $k = \boldsymbol{Q}_p$, and then ρ is a morphism of Lie groups. We let $L(G)$ denote the Lie algebra of G, and we distinguish three cases:

a) $L(G) = \{0\}$. Then G is a countable, discrete, periodic group, and it suffices to show G is locally finite. But since ρ is faithful this follows from a classical result of Schur [8] (see also [9, cor. 4.9]).

b) *$L(G)$ is semi-simple.* Let M be the unique connected algebraic subgroup of $\boldsymbol{GL}_n(\bar{k})$ with Lie algebra $d\rho(L(G)) \otimes_k \bar{k}$. Then M is normalized by $\rho(G)$ and the group $M(k)$ of k-rational points of M is a closed subgroup of $\boldsymbol{GL}_n(k)$ which is a Lie group over k with Lie algebra $d\rho(L(G))$. If H is a compact open subgroup of G, then $\rho(H)$ is a Lie subgroup of $\boldsymbol{GL}_n(k)$ with Lie algebra $d\rho(L(G))$ (recall $k = \boldsymbol{Q}_p$), so $\rho(H) \cap M(k)$ is open in $M(k)$ [6, III, §7, no. 1, th. 2], whence so is $\rho(G) \cap M(k)$. Thus $\rho(G) \cap M(k)$ is closed in $M(k)$, hence in $\boldsymbol{GL}_n(k)$, the group $N = \rho^{-1}(M(k))$ is an open, normal subgroup of G and $\rho \mid N$ is a topological isomorphism of N onto $\rho(G) \cap M(k)$.

Let $\tilde{M}$ denote the normalizer of M in $\boldsymbol{GL}_n(\bar{k})$. There is a k-rational linear representation α of $\tilde{M}$ with kernel M [2, th. 5.6]. Since $\rho(G) \subset \tilde{M}$, the map $\alpha \circ \rho$ yields a faithful representation of G/N, and it follows from a) that G/N is the union of an increasing sequence of finite subgroups. Thus, to complete the proof in case b), it suffices to show that N, or equivalently $\rho(G) \cap M(k)$, is compact. Let $M_1, \cdots, M_t$ be the isotropic simple quotients of M over k and $\pi_i: M \to M_i$ the canonical projection $(1 \leqq i \leqq t)$. The group $L_i = \pi_i(\rho(N))$ is open $(1 \leqq i \leqq t)$; let us show it is compact. By Lemma 5 every element of $\rho(N)$, hence of L_i, lies in a compact group. The group M_i^+ is closed in $M_i(k)$ [6, 6.14], hence $L_i \cap M_i^+$ is an open subgroup of M_i^+ which is the union of its compact subgroups. It is then a proper open subgroup of M_i^+, since the latter contains the k-rational points of a k-split torus and hence an element which generates an infinite discrete subgroup of M_i^+. But it is known that every proper open subgroup of M_i^+ is compact [4, 9.10], hence L_i is compact. It follows that the image of $\rho(N)$ under the mapping

$$\pi = \pi_1 \times \cdots \times \pi_t : M \to M_1 \times \cdots \times M_t$$

is compact. The kernel Q of π is anisotropic over k, hence $Q(k)$ is compact and so is $\rho(N) \cap Q(k)$. Consequently $\rho(N)$ is compact, which completes the proof of the lemma in the case b).

c) *$L(G)$ has a nonzero radical* $\mathfrak{r}$. Let $\mathfrak{a}$ be the algebraic hull of $d\rho(\mathfrak{r})$ and A the unique connected algebraic subgroup of $GL_n(\bar{k})$ with Lie algebra $\mathfrak{a}$. Then A is solvable and normalized by $\rho(G)$, hence $N = \rho^{-1}(A(k))$ is a closed, normal, solvable subgroup of G. Since $\mathfrak{a} \cap d\rho(L(G)) \supset d\rho(\mathfrak{r})$, we have $(d\rho)^{-1}(\mathfrak{a}) \supset \mathfrak{r}$, whence $L(N) = \mathfrak{r}$. Thus G/N is a semi-simple Lie group over k with Lie algebra $L(G)/\mathfrak{r}$. Clearly G/N is σ-compact and the union of its compact subgroups is dense. There exists a k-rational linear representation β of the normalizer B of A in $GL_n(\bar{k})$ with kernel A [2, th. 5.6.]. Then $\beta \circ \rho$ is a faithful, continuous representation of G/N over Q_p. Thus G/N satisfies the hypotheses of our lemma, so by b) it is the union of an increasing sequence of compact open subgroups. This is then also true for G by Lemmas 2 and 3.

THEOREM 1. *Let k be of characteristic zero. Let G be a σ-compact Lie group over k with a finite-dimensional, continuous representation ρ over k such that* $\ker\rho$ *is either the union of an increasing sequence of compact open subgroups or solvable. If G admits an ergodic automorphism, then G is compact.*

PROOF. Let T be an ergodic automorphism of G. If H is a compact open subgroup of G, then $\bigcup_{n \in Z} T^n(H)$ is open, non-empty and invariant under T, hence dense in G. It follows that $G/\ker\rho$ satisfies the hypotheses of Lemma 6, hence $G/\ker\rho$ is the union of an increasing sequence of compact open subgroups. Moreover, if $\ker\rho$ is solvable, it is also such a union by Lemma 3. The theorem now follows from Lemmas 1 and 2.

3. Ergodic automorphisms (k of arbitrary characteristic)

LEMMA 7. *Let G be a reductive, connected linear algebraic group defined over k, and suppose that the derived group $\mathscr{D}G$ of G has k-rank zero. Then $G(k)$ has a unique maximal compact subgroup.*

PROOF. Let S be the maximal k-split torus in the radical R of G, and let D be the group generated by $\mathcal{D}G$ and the maximal anisotropic k-torus in R. Then $G = S \cdot D$, and the canonical mapping $S \times D \to G$ is a central isogeny. Thus $S(k)D(k)$ is a closed cocompact subgroup of $G(k)$ [4, prop. 3.19], and D is anisotropic, so $D(k)$ is compact. Hence $G(k)/S(k)$ is compact.

Let $\chi_1, \cdots, \chi_d \in X(G)_k$ be chosen so that their restrictions to S form a basis of $X(S) \otimes \mathbf{Q}$ ($d = \dim S$). This can be done because the group of restrictions of elements of $X(G)_k$ to S has finite index in $X(S)$. Let $\mathbf{R}_+^*$ denote the multiplicative group of strictly positive real numbers, and define $\rho : G(k) \to (\mathbf{R}_+^*)^d$ by

$$\rho(x) = (|\chi_1(x)|, \cdots, |\chi_d(x)|).$$

The map ρ is a continuous homomorphism whose image is discrete and isomorphic to $\mathbf{Z}^d$. The group $K = \ker \rho$ is an open subgroup of $G(k)$ containing all compact subgroups of $G(k)$. The quotient $K/(S(k) \cap K)$ is then an open, hence compact, subgroup of $G(k)/S(k)$. Since $S(k) \cap K$ is compact, we see that K is compact, which proves the lemma.

THEOREM 2. *Let G be a linear algebraic group defined over k, G^0 its (Zariski) identity component, and suppose that R_uG^0 is also defined over k. Let H be a closed subgroup of $G(k)$ which contains $(G^0)^*$ and admits an ergodic automorphism. Then H is compact.*

PROOF. We shall again show that H is the union of an increasing sequence of compact open subgroups. Since $H \cap G^0$ has finite index in H, we may assume G to be connected (Lemma 2). Let $\pi : G \to G' = G/R_uG$ be the canonical homomorphism. Since $(R_uG)(k) \subset H$ and $\pi(G^*) = G'^+$, we see that $\pi(H)$ is a closed subgroup of $G'(k)$ containing G'^+. By Lemma 5 every element of H lies in a compact subgroup. Then G'^+ contains no elements generating infinite discrete subgroups, hence $G'^+ = \{e\}$. Thus the k-rank of the derived group of G' is zero, so by Lemma 7 $G'(k)$ has a unique maximal compact subgroup K. It follows that $\pi(H) \subset K$. Now, R_uG is trigonalizable over k, so $(R_uG)(k)$ is the union of an increasing sequence of compact open subgroups. The theorem is then proved upon applying Lemmas 1 and 2.

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